2-GENERATED CAYLEY DIGRAPHS ON NILPOTENT GROUPS HAVE HAMILTONIAN PATHS

DAVE WITTE MORRIS

ABSTRACT. Suppose G is a nilpotent, finite group. We show that if $\{a, b\}$ is any 2-element generating set of G, then the corresponding Cayley digraph $\overrightarrow{\mathrm{Cay}}(G; a, b)$ has a hamiltonian path. This implies that all of the connected Cayley graphs of valence ≤ 4 on G have hamiltonian paths.

1. INTRODUCTION

Let G be a group. (All groups are assumed to be finite.)

Definition. For any subset S of G, the *Cayley digraph of S on G* is the directed graph whose vertices are the elements of G, and with a directed edge $g \to gs$, for every $g \in G$ and $s \in S$. It is denoted $\overrightarrow{\mathrm{Cay}}(G; S)$.

It is known that every connected Cayley digraph on G has a hamiltonian path if either G is abelian (see Lemma 2.1) or G has prime-power order (see Theorem 2.2). Since abelian groups and p-groups are the basic examples of nilpotent groups, it is natural to ask whether it suffices to assume that G is nilpotent. We provide some evidence that this may indeed be the case:

Theorem 1.1. *Every connected Cayley digraph of outvalence* 2 *on any nilpotent group has a hamiltonian path.*

There is no need to make any restriction on the outvalence of $\overrightarrow{\mathrm{Cay}}(G; S)$ if we assume the nilpotent group G has only one Sylow subgroup that is not abelian:

Theorem 1.2. *If $G = P \times A$, where P has prime-power order, and A is abelian, then every connected Cayley digraph on G has a hamiltonian path.*

Remark. In abstract terms, the assumption $G = P \times A$ in Theorem 1.2 is equivalent to assuming that G is nilpotent and the commutator subgroup of G has prime-power order.

The above results for directed graphs have the following consequence for (undirected) Cayley graphs.

Corollary 1.3. *Every connected Cayley graph of valence ≤ 4 on any nilpotent group has a hamiltonian path.*

Remark. One can show quite easily that if $S = \{a, b\}$ is a 2-element generating set of a group G, such that $|a| = 2$, $|b| = 3$, and $|G| > 9|ab^2|$, then $\overrightarrow{\mathrm{Cay}}(G; a, b)$ does not have a hamiltonian path. (This observation is attributed to J. Milnor [2, p. 267].) Examples in which G is (super)solvable can be constructed by taking G to be an appropriate semidirect product $\mathbb{Z}_6 \ltimes \mathbb{Z}_p$, where p is a large prime that is congruent to 1 modulo 6. Therefore, the word "nilpotent" cannot be replaced with the word "solvable" (or even "supersolvable") in the statement of Theorem 1.1.

After some preliminaries in Section 2, the above results are proved in Section 3, by using the methods of [5]. See the bibliography of [4] for references on the search for hamiltonian cycles in Cayley graphs on general (non-nilpotent) groups.

2. Preliminaries

All groups in this paper are assumed to be finite.

Notation. Let G be a group, let S be any subset of G, and let H be any subgroup of G.
- The *Cayley digraph* $\overrightarrow{\mathrm{Cay}}(G; S)$ is the directed graph whose vertex set is G, and with an arc from g to gs, for every $g \in G$ and $s \in S$.
- The *Cayley graph* $\mathrm{Cay}(G; S)$ is the (undirected) graph that underlies $\overrightarrow{\mathrm{Cay}}(G; S)$. Thus, its vertex set is G, and g is adjacent to both gs and gs^{-1}, for every $g \in G$ and $s \in S$.
- $H \backslash \overrightarrow{\mathrm{Cay}}(G; S)$ denotes the digraph in which:
  - the vertices are the right cosets of H, and
  - there is a directed edge from Hg to Hgs, for each $g \in G$ and $s \in S$.
- $H^G = \langle g^{-1}hg \mid h \in H, g \in G \rangle$ is the *normal closure* of H in G.
- $\langle S^{-1}S \rangle = \langle s_1^{-1}s_2 \mid s_1, s_2 \in S \rangle$ is the *arc-forcing subgroup*. Note that, for any $a \in S$, we have $\langle S^{-1}S \rangle = \langle a^{-1}S \rangle = \langle a^{-1}s \mid s \in S \rangle$.
- For $s_1, s_2, \ldots, s_n \in S$, we use $(s_i)_{i=1}^n = (s_1, s_2, s_3, \ldots, s_n)$ to denote the walk in $\overrightarrow{\mathrm{Cay}}(G; S)$ that visits (in order) the vertices

$$e, \ s_1, \ s_1s_2, \ s_1s_2s_3, \ \ldots, \ s_1s_2\cdots s_n.$$

Also, $(s_1, s_2, s_3, \ldots, s_n)\#$ denotes the walk $(s_1, s_2, s_3, \ldots, s_{n-1})$ that is obtained by deleting the last term of the sequence.

Terminology. Contrary to most authors, we consider both K_2 and the loop on a single vertex to have hamiltonian cycles. This is because each of these graphs has a hamiltonian path whose terminal vertex is adjacent to its initial vertex.

The following well-known observation is very easy to prove.

Lemma 2.1 ([1, Thm. 30.3, p. 506])**.** *Every connected Cayley digraph on any abelian group has a hamiltonian path.*

In the remainder of this section, we recall some useful results from [5].

Theorem 2.2 (Witte [5])**.** *Every nontrivial, connected Cayley digraph on any group of prime-power order has a hamiltonian cycle.*

Lemma 2.3 (cf. [5, Lem. 4.1])**.** *Suppose H is a subgroup of a group G. If G is nilpotent, then there is a subnormal series*

$$H = H_1 \lhd H_2 \lhd \cdots \lhd H_m = H^G$$

of H^G, such that, for $1 \le k < m$:

(i) *H_{k+1}/H_k is generated by a G-conjugate of H, and*
(ii) *H_{k+1}/H_k has prime-power order.*

Proof. The desired conclusion is proved in [5, Lem. 4.1] under the stronger assumption that G is a p-group. The general result follows from this special case, since every nilpotent group is a direct product of p-groups.

For the reader's convenience, we provide a proof from scratch: given $H_1, \ldots, H_k$, with $H_k \subsetneq H^G$, we show how to construct H_{k+1}. Since $H \subseteq H_k \subsetneq H^G$, we have $H_k \ntriangleleft G$, which means $N_G(H_k) \ne G$. Then, because proper subgroups of nilpotent groups are never self-normalizing [3, Hauptsatz III.2.3(c), p. 260], we have $N_G(H_k) \subsetneq N_G\big(N_G(H_k)\big)$, so we may choose some $x \in G$, such that

$$x \text{ normalizes } N_G(H_k), \text{ but } x \notin N_G(H_k).$$

Since $x \notin N_G(H_k)$, we know $x^{-1}H_k x \nsubseteq H_k$. We also know (from property (i) and induction) that H_k is generated by G-conjugates of H. Hence, there exists $g \in G$, such that

$$g^{-1}Hg \nsubseteq H_k, \text{ and } g^{-1}Hg \subseteq x^{-1}H_k x.$$

Let $H_{k+1} = H_k \cdot (g^{-1}Hg)$. Then:

- H_{k+1} properly contains H, because $g^{-1}Hg \nsubseteq H_k$, and
- $H_k \lhd H_{k+1}$, because $g^{-1}Hg \subseteq x^{-1}H_k x \subseteq N_G(H_k)$ (since x normalizes $N_G(H_k)$).

By construction, the quotient H_{k+1}/H_k is generated by $g^{-1}Hg$.

Since G is nilpotent, it is the direct product of its Sylow subgroups: $G = P_1 \times \cdots \times P_r$. Hence, we may write $g = g_1 g_2 \cdots g_r$, with $g_i \in P_i$. Furthermore,

every subgroup of G is the direct product of its intersections with the Sylow subgroups of G. Therefore, since $g^{-1}Hg \nsubseteq H_k$, there is some i, such that $(g^{-1}Hg) \cap P_i \nsubseteq H_k \cap P_i$. This means $g_i^{-1}Hg_i \nsubseteq H_k$. We also have

$$g_i^{-1}Hg_i \subseteq \langle H, (g_i^{-1}Hg_i) \cap P_i \rangle \subseteq \langle H, g^{-1}Hg \rangle \subseteq \langle H_k, H_{k+1} \rangle = H_{k+1}.$$

Hence, there is no harm in assuming $g = g_i \in P_i$. Then $g^{-1}Hg \subseteq P_iH$, so $|H_{k+1}/H_k|$ is a divisor of $|P_i|$, which is a prime-power. $\qquad\square$

Remark 2.4. The assumption that G is nilpotent in Lemma 2.3 can be replaced with the assumption that H^G is a p-group (for some prime p). To see this, note that if $H_k \ntriangleleft H^G$, then, since H^G is nilpotent, the proof of Lemma 2.3 constructs an appropriate subgroup $H_{k+1} = H_k \cdot (g^{-1}Hg)$, with $g \in H^G$. On the other hand, if $H_k \triangleleft H^G$, then every G-conjugate of H normalizes H_k, so it is easy to construct H_{k+1}. (Since $|H_{k+1}/H_k|$ is a divisor of $|H^G|$, which is a power of p, property (ii) is automatically satisfied.)

Lemma 2.5 ([5, Lem. 5.1]). *Suppose S generates a group G, and let H^+ and H^- be subgroups of G with $H^- \triangleleft H^+$. If*
- *there is a hamiltonian cycle in $H^+\backslash\overrightarrow{\mathrm{Cay}}(G; S)$,*
- *every connected Cayley digraph on H^+/H^- has a hamiltonian cycle, and*
- *H^+/H^- is generated by a G-conjugate of the arc-forcing subgroup $\langle S^{-1}S \rangle$,*

then there is a hamiltonian cycle in $H^-\backslash\overrightarrow{\mathrm{Cay}}(G; S)$.

Essentially the same proof establishes an analogous result for hamiltonian paths, but we need only the following simplified version in which H^- is trivial:

Lemma 2.6 (cf. [5, Lem. 5.1]). *Suppose S generates a group G, and let $H = \langle S^{-1}S \rangle$ be the arc-forcing subgroup. If*
- *there is a hamiltonian cycle in $H\backslash\overrightarrow{\mathrm{Cay}}(G; S)$, and*
- *every connected Cayley digraph on H has a hamiltonian path,*

then there is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G; S)$.

Proof. Let
- $(s_i)_{i=1}^m$ be a hamiltonian cycle in the quotient digraph $H\backslash\overrightarrow{\mathrm{Cay}}(G; S)$, and
- $(s_1 s_2 \cdots s_{m-1} a_i)_{i=1}^{n-1}$ be a hamiltonian path in the Cayley digraph $\overrightarrow{\mathrm{Cay}}(H; s_1 s_2 \cdots s_{m-1} S)$.

Then it is not difficult to verify that

$$(s_1, s_2, \ldots, s_{m-1}, a_i)_{i=1}^n \#$$

is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G; S)$. $\qquad\square$

3. Proofs of the main results

The heart of our argument is contained in the following result, which is adapted from the proof of [5, Thm. 6.1], and may be of independent interest.

Proposition 3.1. *Let*
- *S be a generating set of a finite group G, and*
- *$H = \langle S^{-1}S \rangle$ be the arc-forcing subgroup.*

If
- *G is nilpotent, and*
- *every connected Cayley digraph on H has a hamiltonian path (or hamiltonian cycle, respectively),*

then $\overrightarrow{\mathrm{Cay}}(G; S)$ has a hamiltonian path (or hamiltonian cycle, respectively).

Proof. Consider the subnormal series

$$H = H_1 \lhd H_2 \lhd \cdots \lhd H_m = H^G$$

that is provided by Lemma 2.3, and choose some $a \in S$. Since

$$\langle a, H \rangle = \langle a, a^{-1}S \rangle = \langle S \rangle = G,$$

we know that a generates the quotient group $G/H^G = G/H_m$. Thus, $H_m \backslash \overrightarrow{\mathrm{Cay}}(G; a)$ is a directed cycle. Furthermore, for each k, Theorem 2.2 tells us that every connected Cayley digraph on H_{k+1}/H_k has a hamiltonian cycle. Thus, repeated application of Lemma 2.5 (with $H^+ = H_{k+1}$ and $H^- = H_k$, for $k = m - 1, m - 2, \ldots, 1$) tells us that $H_1 \backslash \overrightarrow{\mathrm{Cay}}(G; S)$ has a hamiltonian cycle. Since $H_1 = H$, this means $H \backslash \overrightarrow{\mathrm{Cay}}(G; S)$ has a hamiltonian cycle.

By assumption, we also know that every connected Cayley digraph on H has a hamiltonian path (or hamiltonian cycle, respectively). Therefore, Lemma 2.6 (or Lemma 2.5 with $H^+ = H$ and $H^- = \{e\}$) provides a hamiltonian path (or hamiltonian cycle) in $\overrightarrow{\mathrm{Cay}}(G; S)$. □

Remark. The proof of Theorem 2.2 is a minor modification of the proof of Proposition 3.1. Namely, rather than appealing to Theorem 2.2 in order to know that every connected Cayley digraph on H_{k+1}/H_k has a hamiltonian cycle, one can assume this is true by induction on $|G|$. The same induction hypothesis also implies that every connected Cayley digraph on H has a hamiltonian cycle.

Corollary 3.2. *Let S be a generating set of the group G. If*
- *G is nilpotent, and*
- *the arc-forcing subgroup $H = \langle S^{-1}S \rangle$ is abelian,*

then $\overrightarrow{\mathrm{Cay}}(G; S)$ has a hamiltonian path.

Proof. Lemma 2.1 tells us that every connected Cayley digraph on H has a hamiltonian path, so Proposition 3.1 applies. □

Proof of Theorem 1.1. Let $\{a, b\}$ be a 2-element generating set for G. Then the arc-forcing subgroup $H = \langle a^{-1}b \rangle$ is cyclic, so it is abelian. Therefore Corollary 3.2 provides a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G; a, b)$. $\square$

Proof of Theorem 1.2. Let $\overrightarrow{\mathrm{Cay}}(G; S)$ be a connected Cayley digraph on $G = P \times A$, and let $H = \langle S^{-1}S \rangle$ be the arc-forcing subgroup. We may assume the generating set S is minimal.

Case 1. Assume $H \neq G$. By induction on $|G|$, we may assume every connected Cayley digraph on H has a hamiltonian path. Then Proposition 3.1 provides a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G; S)$.

Case 2. Assume $H = G$. Choose some $a \in S$, and let $^-: G \to P$ be the natural projection homomorphism. Since $G = H = \langle a^{-1}S - \{e\} \rangle$, and the minimal generating sets of any finite p-group all have the same cardinality [3, Satz III.3.15, p. 273], there is a proper subset S_0 of S, such that $\langle \overline{S_0} \rangle = P$. Since $G/P \cong A$ is abelian, this implies $[G, G] \subseteq \langle S_0 \rangle$. Therefore $\langle S_0 \rangle \trianglelefteq G$.

Let $N = \langle S_0 \rangle \trianglelefteq G$. Since S_0 is a proper subset of S, and S is minimal, we know $N \neq G$. Also, we may assume $[G, G]$ is nontrivial, for otherwise Lemma 2.1 provides a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G; S)$. Therefore N is nontrivial. Hence, by induction on $|G|$, we may assume every connected Cayley digraph on N or G/N has a hamiltonian path; let

- $(s_i)_{i=1}^n$ be a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(N; S_0)$, and
- $(t_j)_{j=1}^q$ be a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G/N; S)$.

Then it is easy to see (and well known) that $\left((s_i)_{i=1}^n, t_j \right)_{j=1}^{q+1} \#$ is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G; S)$. $\square$

Proof of Corollary 1.3. Suppose $\mathrm{Cay}(G; S)$ is a connected Cayley graph of valence ≤ 4, and G is nilpotent. There is no harm in assuming that the generating set S is minimal. Let S_2 be the set of elements of order 2 in S. Also, let P be the Sylow 2-subgroup of G, so $G = P \times K$, where $|K|$ is odd.

If $\#S - \#S_2 \leq 1$, then, since $S_2 \subseteq P$, we know $K \cong G/P$ is cyclic. Therefore K is abelian, so Theorem 1.2 applies.

We may now assume $\#S - \#S_2 \geq 2$. Then

$$4 \geq \text{valence of } \mathrm{Cay}(G; S) = \#(S \cup S^{-1})$$
$$= 2(\#S - \#S_2) + \#S_2 \geq 2(\#S - \#S_2) \geq 2 \cdot 2.$$

We must have equality throughout, so $\#S = 2$ (and $S_2 = \emptyset$). Then Theorem 1.1 provides a hamiltonian path in $\mathrm{Cay}(G; S)$. $\square$

The following generalization of Theorem 2.2 is sometimes useful.

Corollary 3.3. *Suppose*
- *S is a nonempty generating set of a group G,*
- *N is a normal p-subgroup of G, for some prime p, and*
- *there exists $a \in G$, such that $S \subset aN$.*

Then $\overrightarrow{\text{Cay}}(G; S)$ *has a hamiltonian cycle.*

Proof. Let $H = \langle S^{-1}S \rangle \subset (aN)^{-1}(aN) = N$. Since $N \lhd G$, this implies $H^G \subset N$, so H^G is a p-group. Hence, Remark 2.4 provides a subnormal series as in Lemma 2.3, and Theorem 2.2 tells us that every connected Cayley digraph on H has a hamiltonian cycle. Then the proof of Proposition 3.1 provides a hamiltonian cycle in $\overrightarrow{\text{Cay}}(G; S)$. $\qquad\square$

References

1. J. A. Gallian: *Contemporary Abstract Algebra, 7th edition,* Houghton Mifflin, Boston, 2010.
2. W. Holsztyński and R. F. E. Strube: Paths and circuits in finite groups, *Discrete Math.* 22 (1978), no. 3, 263–272.
3. B. Huppert: *Endliche Gruppen I.* Springer, New York, 1967.
4. K. Kutnar, D. Marušič, J. Morris, D. W. Morris, and P. Šparl: Hamiltonian cycles in Cayley graphs whose order has few prime factors, *Ars Math. Contemp.* (to appear). http://arxiv.org/abs/1009.5795
5. D. Witte: Cayley digraphs of prime-power order are hamiltonian. *J. Comb. Th. B* 40 (1986) 107–112.

Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, T1K 3M4, Canada
E-mail address: Dave.Morris@uleth.ca, http://people.uleth.ca/∼dave.morris/